1301. Using the compound-angle formula,

$$\begin{aligned}&\cos x\cos 30° - \sin x \sin 30° = \sin x\\ \Longrightarrow\ &\tfrac{\sqrt{3}}{2}\cos x - \tfrac{1}{2}\sin x = \sin x\\ \Longrightarrow\ &\sqrt{3}\cos x = 3\sin x.\end{aligned}$$

Dividing by $3\cos x$,

$$\begin{aligned}\tfrac{\sqrt{3}}{3} &= \frac{\sin x}{\cos x}\\ \Longrightarrow\ \tan x &= \tfrac{\sqrt{3}}{3}.\end{aligned}$$

Therefore, $x = \arctan\frac{\sqrt{3}}{3} = 30°$ or, diametrically on a unit circle, $210°$.

1302. (a) $\left.\dfrac{dE}{dt}\right|_{t=0} = 0.$

(b) i. For E to be increasing, its derivative must be positive:

$$2.15616t - 0.088210t^2 > 0.$$

The LHS is a negative quadratic; its roots are $t = 0$ and $t = 24.443487...$. So, E is increasing for $t \in (0, 24.443)$, to 5sf.

ii. For the rate of change of energy $\frac{dE}{dt}$ to be increasing, we need the second derivative to be positive:

$$\frac{d^2E}{dt^2} = 2.15616 - 0.17642t > 0.$$

This is a linear inequality, whose solution is $t < 12.2217435....$ We are told that $t \geq 0$, so $t \in [0, 12.222)$, to 5sf.

1303. If events A and B are mutually exclusive, then knowledge of A's occurrence rules B out entirely, i.e. $\mathbb{P}(B \mid A) = 0$. But since $\mathbb{P}(B) \neq 0$, this means that $\mathbb{P}(B \mid A) \neq \mathbb{P}(B)$, which is the definition of dependence.

1304. (a) $2^{\log_2 x + \log_2 y} \equiv 2^{\log_2 x} \times 2^{\log_2 y} \equiv xy$,

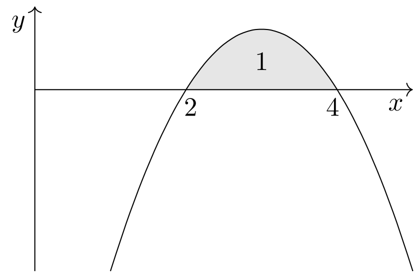
(b) $e^{\ln x + \ln y} \equiv e^{\ln x} \times e^{\ln y} \equiv xy.$

1305. Splitting the octagon into eight sectors, each has area $\frac{1}{4}(1+\sqrt{2})$. Each is an isosceles triangle with $45°$ subtended at the centre, between two radii r. Using the sine area formula, we know that

$$\begin{aligned}&\tfrac{1}{2}r^2\sin 45° = \tfrac{1}{4}(1+\sqrt{2})\\ \Longrightarrow\ &\tfrac{1}{4}\sqrt{2}r^2 = \tfrac{1}{4}(1+\sqrt{2})\\ \Longrightarrow\ &r^2 = \tfrac{\sqrt{2}}{2} + 1.\end{aligned}$$

Therefore $r = \sqrt{\frac{\sqrt{2}}{2} + 1}$.

1306. The quadratic crosses the x axis at 2 and 4, and, since the integral is positive, must be above the x axis between those points. So, it's a negative parabola:



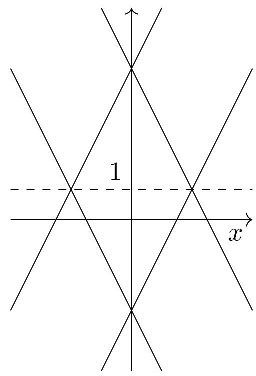
1307. (a) The volume of the pipe is $50 \times 20 = 1000$ cm^3. The water in the pipe has mass 1 kg.

(b) The water accelerates from $u = 0$ to $v = 2$ with displacement $s = 0.5$. Using $v^2 = u^2 + 2as$,

$$\begin{aligned}4 &= 2a \times 0.5\\ \Longrightarrow a &= 4 \text{ ms}^{-2}.\end{aligned}$$

(c) Since the water has a consistent acceleration, it can be modelled as a single object of mass 1 kg. NII is $F = 4 \times 1 = 4$ N.

1308. Sketching:



The shape has the lines $x = 0$ and $y = 1$ as lines of symmetry. Hence, it is a rhombus.

1309. (a) By definition, the average deviation from the mean is 0.

(b) The mean value of the first term $(x_i - \bar{x})$ is the variance, by definition in the formula

$$\frac{\sum(x_i - \bar{x})^2}{n} = s_x^2.$$

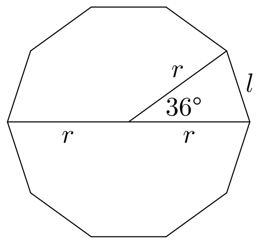
Subtracting the constant value s_x^2 from each gives a mean value of 0.

1310. (a) No. A counterexample is $\mathrm{f}(x) = x(x-1)$ and $\mathrm{g}(x) = x(x-1)$, with $a = 0, b = 0, c = 1$.

(b) No. A counterexample is $\mathrm{f}(x) = x(x-1)$ and $\mathrm{g}(x) = x(x-1)$, with $a = 0, b = 1, c = 0$.

1311. Each equilateral face has area $\frac{1}{2} \times 1 \times \frac{\sqrt{3}}{2}$, and the square base has area 1, so the total surface area is $\sqrt{3} + 1$.

1312. Decagon:



Splitting the isosceles triangle shown in two, we have $\frac{l}{2} = r \sin 18°$. This gives, as required,

$$d = 2r = \frac{l}{\sin 18°} = l \operatorname{cosec} 18°.$$

1313. Friction does act to oppose (any possible) motion, but such motion is the relative motion between two objects, not the absolute motion of one object. If a car brakes to a halt on a road, the NIII pair is

- friction acting backwards on the car, and
- friction acting forwards on the road.

These both act to oppose the motion of the car relative to the road.

1314. Carrying out the definite integrals,

$$\begin{aligned} & \int_0^1 12x^2\,dx = \int_0^k x^3\,dx \\ \Longrightarrow\ & \Big[4x^3\Big]_0^1 = \Big[\tfrac{1}{4}x^4\Big]_0^k \\ \Longrightarrow\ & 4 - 0 = \tfrac{1}{4}k^4 - 0 \\ \Longrightarrow\ & 16 = k^4 \\ \Longrightarrow\ & k = \pm 2. \end{aligned}$$

1315. (a) The range is $(0, \infty)$.
(b) Negating the inputs leaves the range as $(0, \infty)$.
(c) Negating the outputs gives $(-\infty, 0)$.

1316. The new parabola is a reflection of the old in the y axis. To enact this reflection, we replace x with $-x$, giving

$$\begin{aligned} & y = (-x)^2 + p(-x) + q \\ \Longrightarrow\ & y = x^2 - px + q. \end{aligned}$$

1317. (a) This is false: $\text{lcm}(4, 6) = 12$.
(b) This is true by definition.
(c) This is false: $\text{lcm}(1, 4) = 4$, but 1 and 4 are not primes.

1318. The equation for intersections of curve and normal $x^4 - x^2 = mx + c$ is quartic. It has a single root at $x = 1$, because the normal crosses the curve there. Hence, the quartic has exactly one factor of $(x-1)$. This leaves a cubic factor. Every cubic has a root, so there must be at least one other intersection.

1319. The implication is $x \in [a, b] \Longleftarrow x \in (a, b)$. This is because, if $a < x < b$, then $a \le x \le b$. The boundary values $x = a, b$ are counterexamples to the forwards implication.

1320. The gradients of the lines are $-2/p$ and $1/p+1$. Since the lines are perpendicular, these multiply to -1:

$$\begin{aligned} & -\frac{2}{p} \times \frac{1}{p+1} = -1 \\ \Longrightarrow\ & 2 = p(p+1) \\ \Longrightarrow\ & p^2 + p - 2 = 0 \\ \Longrightarrow\ & p = -2, 1. \end{aligned}$$

1321. (a) $\{1, 2, 3, ...\} \cap [-2, 2] = \{1, 2\}$,
(b) $\{0, -1, -2, ...\} \cap (-3, 3] = \{-2, -1\}$,
(c) $\{2, -2, 3, -3, ...\} \cap (-4, 4) = \{-3, -2, 2, 3\}$.

1322. Any AP can be expressed as $u_n = a + (n-1)d$. If f has constant first derivative, then $f'(x) = k$, so $f(x) = kx + c$. Substituting,

$$\begin{aligned} f(u_n) &= ku_n + c \\ &= k(a + (n-1)d) + c \\ &= ka + c + (n-1)kd. \end{aligned}$$

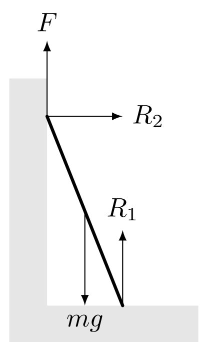
This is the ordinal formula for an AP with first term $ka + c$ and common difference kd. □

1323. This is a quadratic in t^3. Using the formula,

$$\begin{aligned} & pt^6 + qt^3 + r = 0 \\ \Longrightarrow\ & t^3 = \frac{-q \pm \sqrt{q^2 - 4pr}}{2p} \\ \Longrightarrow\ & t = \sqrt[3]{\frac{-q \pm \sqrt{q^2 - 4pr}}{2p}}. \end{aligned}$$

1324. The conclusion is too strong. A hypothesis test never tells you that a null hypothesis *is* definitely false, only that you have sufficient evidence (here at the 5% significance level) to reject it. Rejecting it, as opposed to saying it is false, leaves open the possibility that it is, in fact, true. Which it might well be!

1325. Assume, for a contradiction, that the ladder is in equilibrium. The forces are as follows:



Consider moments around the foot of the ladder. Since the weight has an anticlockwise moment, R_2 cannot be zero (F would then be zero as well). R_2 is the only horizontal force, so there is horizontal acceleration. This is a contradiction. Equilibrium cannot be maintained. □

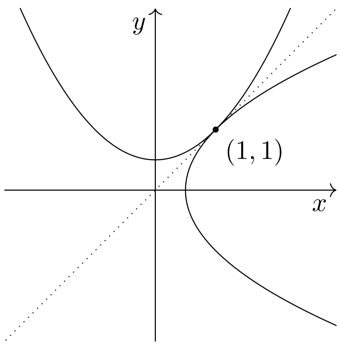
1326. Using $\sin^2 x + \cos^2 x \equiv 1$,

$$
\begin{aligned}
& 2 - 2\cos^2 x + \cos x = 3 \\
\Longrightarrow\ & 2\cos^2 x - \cos x + 1 = 0.
\end{aligned}
$$

This is a quadratic in $\cos x$. Its discriminant is $\Delta = -7 < 0$. So, the equation has no real roots.

1327. Rewriting $\frac{\pi i}{20}$ as $\frac{2\pi i}{40}$, we can see that each i step involves a rotation around the point $(2, 4)$ by $\frac{2\pi i}{40}$ radians. Since there are 2π radians at the centre, the polygon, centred at $(2, 4)$ has 40 sides.

1328. As the roles of x and y switch, the two curves are reflections in $y = x$. Hence, they intersect where $y = x$. Substituting $y = x$ into the first equation, $2x = x^2 + 1$, which is $(x - 1)^2 = 0$. Since $x = 1$ is a double root, the line $y = x$ is tangent to the first curve. Therefore, the curves must be tangent to each other at $(1, 1)$.



1329. The graph $y = \mathrm{h}(x)$ is stationary at $(3, -1)$, since $\mathrm{h}'(3) = 0$ and $\mathrm{h}(3) = -1$. Furthermore, this is a local maximum, since

$$\mathrm{h}''(3) = -1 < 0.$$

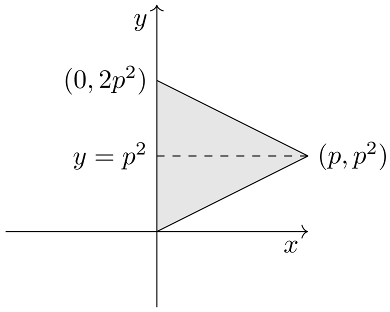
But, for a quadratic, a local maximum is a global maximum, so $\mathrm{h}(x) \le -1$. The equation $\mathrm{h}(x) = 0$, therefore, has no real roots.

1330. Using a log law, we have $\ln\big(x(4+x)\big) = \ln 2$. Then, exponentiating both sides gives

$$
\begin{aligned}
& x(4 + x) = 2 \\
\Longrightarrow\ & x^2 + 4x - 2 = 0 \\
\Longrightarrow\ & x = \pm\sqrt{6} - 2.
\end{aligned}
$$

But the lower root $-\sqrt{6} - 2 < 0$ cannot be input into the natural logarithm function. Hence, the solution is $x = \sqrt{6} - 2$.

1331. The scenario is



Treating the y axis as the base, the area is

$$
\begin{aligned}
A_\triangle &= \tfrac{1}{2}bh \\
&= \tfrac{1}{2} \times 2p^2 \times p \\
&\equiv p^3, \text{ as required.}
\end{aligned}
$$

1332. Since $\mathrm{g}(x)$ is a cubic, $\mathrm{g}'(x)$ must be a quadratic. It has roots at ± 2, and is a positive parabola, so its equation is

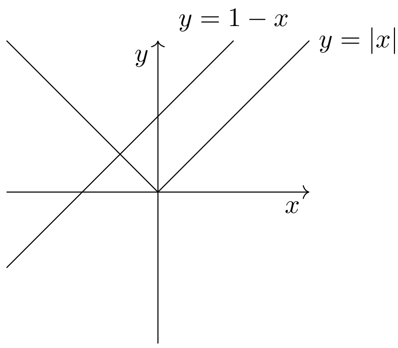
$$
\begin{aligned}
\mathrm{g}'(x) &= a(x + 2)(x - 2) \\
&\equiv ax^2 - 4a.
\end{aligned}
$$

Integrating this gives

$$\mathrm{g}(x) = \tfrac{1}{3}ax^3 - 4ax + b,$$

where $a > 0$ and b are constants.

1333. Rearranging to $|x| > 1 - x$, we compare $y = |x|$ and $y = 1 - x$. These are

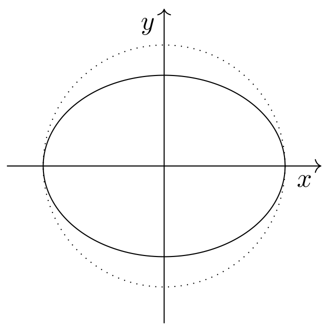


The intersection is at $x = -1/2$, and we need the mod graph to be above the straight line. Hence, $x \in (-\infty, -1/2)$.

1334. Separating the variables,

$$
\begin{aligned}
& \operatorname{cosec} x \frac{dy}{dx} - \tan y = 1 \\
\Longrightarrow\ & \operatorname{cosec} x \frac{dy}{dx} = 1 + \tan y \\
\Longrightarrow\ & \frac{dy}{dx} = \sin x(1 + \tan y) \\
\Longrightarrow\ & \frac{1}{1 + \tan y}\frac{dy}{dx} = \sin x.
\end{aligned}
$$

1335. The given curve is an ellipse centred on the origin, with axis intercepts at $(\pm 15, 0)$ and $(0, \pm 20)$.



So, the longest radius, along the x axis, has length 20. The smallest circle (dotted above) enclosing the ellipse must have radius 20, and area 400π.

1336. According to the (reverse) chain rule, a scale factor $1/2$ should emerge. Replacing x by $2x$ is a stretch scale factor $1/2$ in the x direction, reducing areas by $1/2$. The corrected version is

$$\int (2x+1)^2 \, dx = \tfrac{1}{6}(2x+1)^3 + c.$$

— Nota Bene —

Replacing x by $2x$ stretches in the x direction by scale factor $\frac{1}{2}$. This is a compression. It

- scales gradients by 2, making every tangent twice as steep as it was before,
- scales areas by $1/2$, making every area twice as narrow as it was before.

These are inverses: finding areas (integration) is the inverse of finding gradients (differentiation).

1337. Splitting into vertical and horizontal components:

s_y	0	s_x	s_x
u_y	$70\sin 15°$	u_x	$70\cos 15°$
v_y		v_x	
a_y	$-g$	a_x	0
t	t	t	t

Vertically, $0 = (70\cos 15°)t - \frac{1}{2}gt^2$. Solving gives $t = 0$ (launch) or $t = 140\cos 15°/g$ (landing). Hence, the range is

$$s_x = \frac{140\cos 15°}{g} \times 70\sin 15° = 250 \text{ m}.$$

1338. Multiplying both sides by $(1+x)^2$,

$$\begin{aligned} & 1 + x + 1 = 2(1+x)^2 \\ \Longrightarrow\ & 2x^2 + 3x = 0 \\ \Longrightarrow\ & x(2x+3) = 0 \\ \Longrightarrow\ & x = 0 \text{ or } x = -\tfrac{3}{2}. \end{aligned}$$

1339. We know that, since $r \neq 1$,

$$a(1-r)^2 > 0.$$

Multiplying out, this is

$$\begin{aligned} & a - 2ar + ar^2 > 0 \\ \Longleftrightarrow\ & a + ar^2 > 2ar. \end{aligned}$$

If a, b, c are in GP, then $ar = b$ and $ar^2 = c$, giving us the result.

1340. (a) Scaling $\frac{4}{15}$ of the sample back up, while leaving the rest unchanged, we get

$$13.6\left(\tfrac{4}{15} \cdot \tfrac{1}{0.84} + \tfrac{11}{15}\right) = 14.29... = 14.3 \text{ (3sf)}.$$

(b) This is only an estimate because it is possible that e.g. only the heaviest or only the lightest octopi were under-measured.

1341. Using log rules,

$$\begin{aligned} & \ln\frac{1}{e^x} + \ln(2e^x) \\ =\ & \ln e^{-x} + \ln 2 + \ln e^x \\ =\ & -x + \ln 2 + x \\ =\ & \ln 2. \end{aligned}$$

1342. The gradient of the line is $\frac{d-b}{c-a}$, hence, its equation, using $y - y_1 = m(x - x_1)$, is

$$y - b = \frac{d-b}{c-a}(x-a).$$

Setting y to zero for the x intercept:

$$\begin{aligned} & -b = \frac{d-b}{c-a}(x-a) \\ \Longrightarrow\ & \frac{-bc+ab}{d-b} = x - a \\ \Longrightarrow\ & \frac{ab-ad}{b-d} + \frac{bc-ab}{b-d} = x \\ \Longrightarrow\ & x = \frac{bc-ad}{b-d}, \text{ as required.} \end{aligned}$$

1343. (a) This is a gradient calculated using values $x+h$ and $x-h$. The function is $\text{f}(x) = \sqrt{x+1}$.

(b) This is a gradient calculated using points at a and b. The function is $\text{g}(x) = x^2 + \frac{1}{x}$.

1344. Equating the differences,

$$\begin{aligned} & s^2 - s = 6 \\ \Longrightarrow\ & s^2 - s - 6 = 0 \\ \Longrightarrow\ & (s-3)(s+2) = 0 \\ \Longrightarrow\ & s = -2, 3. \end{aligned}$$

In each case, the common difference is 6. So, the possible values for the hundredth term are

$$\begin{aligned} u_{100} &= -2 + 99 \cdot 6 = 592, \\ u_{100} &= 3 + 99 \cdot 6 = 597. \end{aligned}$$

1345. Assume that each pupil has (independently) a $\frac{7}{365}$ probability of having a birthday in a given week.

$$\begin{aligned}&\mathbb{P}(\text{at least one birthday})\\ &= 1 - \mathbb{P}(\text{no birthdays})\\ &= 1 - \left(\tfrac{358}{365}\right)^{25}\\ &= 0.38375...\\ &= 0.384 \text{ (3sf)}.\end{aligned}$$

1346. Multiplying up gives a difference of two squares on the RHS. This simplifies to

$$\begin{aligned}&(1-\sqrt{x})^2 + (1+\sqrt{x})^2 = (1-x)^2\\ \Longrightarrow\ & 2 + 2x = 1 - 2x + x^2\\ \Longrightarrow\ & x^2 - 4x - 1 = 0\\ \Longrightarrow\ & x = \frac{4 \pm \sqrt{16+4}}{2}\\ \Longrightarrow\ & x = 2 \pm \sqrt{5}.\end{aligned}$$

1347. Using a calculator, $2x^3 - 5x^2 - 21x + 36 = 0$ has solution $x = -3, 4, 3/2$. So, by the factor theorem, the factorisation is $(x+3)(x-4)(2x-3)$.

1348. (a) The right-hand function (nearer the input) is performed first, so the domain of fg is B and of gf is A.

(b) The function gf : $A \mapsto A$ cycles the elements of A, mapping each to the one below it. Hence, the sequence given is periodic, with period 3.

1349. Marking outcomes in the possibility space:

	1	2	3	4	5	6
1						
2	✓					
3	✓	✓				
4	✓	✓	✓			

The probability is $\frac{6}{24} = \frac{1}{4}$.

1350. We start with the first Pythagorean trig identity, and then divide through by $\cos^2\theta$:

$$\begin{aligned}&\sin^2\theta + \cos^2\theta \equiv 1\\ \Longrightarrow\ & \tan^2\theta + 1 \equiv \sec^2\theta\\ \Longrightarrow\ & \tan^2\theta \equiv \sec^2\theta - 1, \text{ as required.}\end{aligned}$$

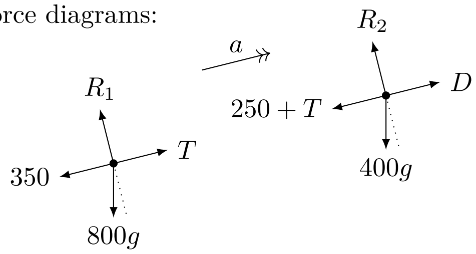
— Nota Bene —

You might wonder whether a proof such as the one above should include a proof of

$$\sin^2\theta + \cos^2\theta \equiv 1.$$

There is no hard-and-fast answer to this, but a rule of thumb. When proving, you aren't trying to rebuild maths from the ground up. Rather, you're trying to pinpoint why this specific result is true. So, *results more basic than the one in question are generally taken as read.* The connotation being, if you're well enough versed in mathematics (either writing or reading) to understand the derivation of the second Pythagorean trig identity from the first, then you probably know why the first Pythagorean trig identity holds.

1351. (a) Force diagrams:



(b) The minimal acceleration to move indefinitely is $a = 0$. Resolving up the slope for the entire system, we get $D - 1200g\sin 5° - 600 = 0$. So, the minimal driving force is $D = 1625$ N (4sf).

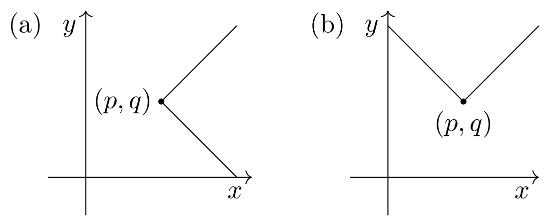
(c) Considering now only the caravan, with $a = 0$, we have $T - 800g\sin 5° - 350 = 0$, which gives the tension in the towbar as $T = 1033$ N (4sf).

1352. Multiplying up by $x^{\frac{1}{2}}$ to eliminate the negative power, $1 + x^{\frac{1}{2}} = x$ is a quadratic in $x^{\frac{1}{2}}$.

$$\begin{aligned}&x - x^{\frac{1}{2}} - 1 = 0\\ \Longrightarrow\ & x^{\frac{1}{2}} = \tfrac{1\pm\sqrt{5}}{2}.\end{aligned}$$

Since $x^{\frac{1}{2}}$ is non-negative, $x = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$.

1353. The sketches are



1354. The possibility space is the interior of the octagon. Calling the side length 1, we can calculate the area of the octagon as

$$\underbrace{1}_{\text{square}} + \underbrace{4 \times \tfrac{\sqrt{2}}{2}}_{\text{rectangles}} + \underbrace{4 \times \tfrac{1}{2}\left(\tfrac{\sqrt{2}}{2}\right)^2}_{\text{triangles}} = 2 + 2\sqrt{2}.$$

Hence, the probability is $\dfrac{1}{2+2\sqrt{2}}$.

1355. (a) True.

(b) True. It's the same statement as in (a).

(c) Not true (unless $y = x$ happens to be true). If y is constant then the LHS is zero.

1356. Starting with the LHS,

$$\begin{aligned} & T_{n+1}+T_n \\ \equiv\ & \tfrac{1}{2}(n+1)(n+2)+\tfrac{1}{2}n(n+1) \\ \equiv\ & \tfrac{1}{2}(n+1)\big((n+2)+n\big) \\ \equiv\ & \tfrac{1}{2}(n+1)(2n+2) \\ \equiv\ & (n+1)^2. \end{aligned}$$

Starting with the RHS,

$$\begin{aligned} & (T_{n+1}-T_n)^2 \\ \equiv\ & \left(\tfrac{1}{2}(n+1)(n+2)-\tfrac{1}{2}n(n+1)\right)^2 \\ \equiv\ & \left(\tfrac{1}{2}n^2+\tfrac{3}{2}n+1-\tfrac{1}{2}n^2-\tfrac{1}{2}n\right)^2 \\ \equiv\ & (n+1)^2. \end{aligned}$$

Therefore, the identity holds.

1357. By multiplying out and splitting the fraction up, we can simplify the integrand to $x^{-\frac{1}{3}}+2+x^{\frac{1}{3}}$. Then, integrating,

$$\begin{aligned} & \int_0^{1000} x^{-\frac{1}{3}}+2+x^{\frac{1}{3}}\,dx \\ =\ & \left[\tfrac{3}{2}x^{\frac{2}{3}}+2x+\tfrac{3}{4}x^{\frac{4}{3}}\right]_0^{1000} \\ =\ & (150+2000+7500)-(0) \\ =\ & 9650, \text{ as required.} \end{aligned}$$

1358. The gradient $m_{AB}=-\frac{1}{3}$, so the perp. bisector is

$$y+1=3(x-4).$$

Then, $m_{AC}=\frac{1}{2}$, so the bisector is

$$y-4=-2(x-9).$$

Solving simultaneously gives the centre $O:(7,8)$. The squared distance $|OA|^2$ is then $36+64=100$. So, the circle has equation $(x-7)^2+(y-8)^2=100$.

1359. The numerator is a difference of two squares. So,

$$\frac{1-9x^{\frac{5}{2}}}{1-3x^{\frac{5}{4}}}\equiv\frac{\left(1-3x^{\frac{5}{4}}\right)\left(1+3x^{\frac{5}{4}}\right)}{1-3x^{\frac{5}{4}}}\equiv 1+3x^{\frac{5}{4}}.$$

1360. Performing the operation $\frac{d}{dx}$, we get $8x+\frac{dy}{dx}=0$. Hence, $\frac{dy}{dx}=-8x$.

1361. (a) Initially, 1 exerts more force, so must gain. We define a in 1's direction. NII gives

$$1480-20t-(1460-10t)=200a.$$

This simplifies to $a=\frac{1}{10}-\frac{1}{20}t$. Integrating, we get $v=\frac{1}{10}t-\frac{1}{40}t^2+c$. The $+c$ is zero, since the game starts from rest. Integrating again gives $s=\frac{1}{20}t^2-\frac{1}{120}t^3+d$. Again, since the initial displacement is zero, $d=0$.

The maximum positive displacement occurs when $v=0$. This is at $t=0$ or $t=4$. At that point, the displacement is $s=\frac{4}{15}<1$, so competitor 1 does not win.

(b) Competitor 2 wins when $\frac{1}{20}t^2-\frac{1}{120}t^3=-1$. Using a polynomial solver, $t=7.91532....$ So, competitor 2 wins after approx. 7.9 seconds.

1362. We get two quadratics: $x^2-3x=\pm(3x-1)$. These simplify to $x^2-1=0$ and $x^2-6x+1=0$. So, the full solution is $x=\pm 1$ or $x=3\pm 2\sqrt{2}$.

1363. The first statement is not true. Any pair (θ,ϕ) such that $\theta-\phi=\pi$ radians is a counterexample.

1364. The possibility space is:

	1	2	3	4	5	6
1				✓	✓	✓
2					✓	✓
3						✓
4	✓					
5	✓	✓				
6	✓	✓	✓			

So, the probability is $\frac{12}{36}=\frac{1}{3}$.

1365. The factor theorem tells us $(15x-4)$ is a factor. Extracting it, we reach $(15x-4)(3x^2-x-2)$. The quadratic factorises: $(15x-4)(3x+2)(x-1)$.

— Nota Bene —

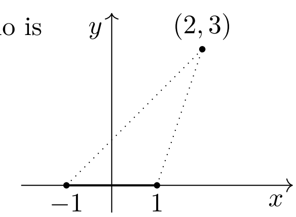
Polynomial long division:

$$\begin{array}{r|l} & 3x^2 \quad -x-2 \\ \hline 15x-4 & 45x^3-27x^2-26x+8 \\ & -45x^3+12x^2 \\ \hline & \qquad -15x^2-26x \\ & \qquad 15x^2-4x \\ \hline & \qquad\qquad -30x+8 \\ & \qquad\qquad 30x-8 \\ \hline & \qquad\qquad\qquad 0 \end{array}$$

1366. Algebraically, we need the squared distance from $(t,0)$ to $(2,3)$ to be 10. Hence, $(t-2)^2+3^2=10$, so $t=1,3$. The latter value is not a valid t value for the line segment, but the former is. So, there is exactly one point, the endpoint $(1,0)$.

— Alternative Method —

The scenario is



By Pythagoras, the lengths of the dotted lines are $\sqrt{18}$ and $\sqrt{10}$. So, there is one point, $(1,0)$.

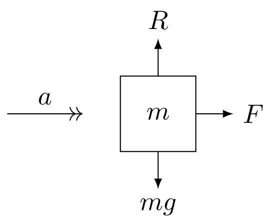
1367. Applying the differential operator,

$$\begin{aligned}\frac{d}{dt}\left(4x+2t^2\right) &= 6t\\ \Longrightarrow 4\frac{dx}{dt}+4t &= 6t\\ \Longrightarrow 4\frac{dx}{dt} &= 2t\\ \Longrightarrow \frac{dx}{dt} &= \tfrac{1}{2}t.\end{aligned}$$

1368. The rectangle divides the cube in half by volume. Hence, placing the first point anywhere (without loss of generality), the probability that the other two are in its half is $\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$.

1369. These are the statements in the factor theorem, whose implication goes both ways, so $\Longleftrightarrow$.

1370. (a) A force diagram for the box is

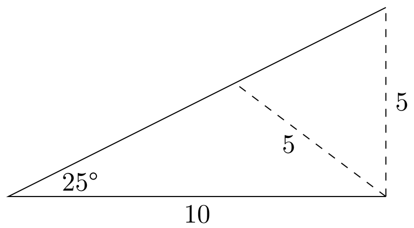


Vertically, $R = mg$, so $F_{\text{max}} = \mu mg$. Maximal acceleration means maximal friction, so NII horizontally is

$$\begin{aligned}\mu mg &= m_{\text{max}}\\ \Longrightarrow a_{\text{max}} &= \mu g.\end{aligned}$$

(b) The direction doesn't matter, so $a_{\text{max}} = \mu g$.

1371. (a) The triangles are as follows:



(b) i. In the sine rule for B, there are two angles B in $(0, 180°)$ satisfying $\sin B = 2\sin 25°$.

ii. In the cosine rule for length c, the quadratic $25 = 100 + c^2 - 20c\cos 25°$ has two roots.

1372. This can be viewed either as a geometric series, or, equivalently, as the recurring decimal 0.99999.... Hence, the sum to infinity is 1.

1373. Since the expectation is $np = 5/3$, the modal values must be 1 and 2. The probabilities are

$$\begin{aligned}\mathbb{P}(X=1) &= {}^5\mathrm{C}_1 \tfrac{1}{3}^1 \tfrac{2}{3}^4 = \tfrac{80}{243}\\ \mathbb{P}(X=2) &= {}^5\mathrm{C}_2 \tfrac{1}{3}^2 \tfrac{2}{3}^3 = \tfrac{80}{243}.\end{aligned}$$

1374. (a) The graph is concave everywhere, so any SP must be a maximum. For a continuous graph, it is not possible to have two local maxima without a local minimum.

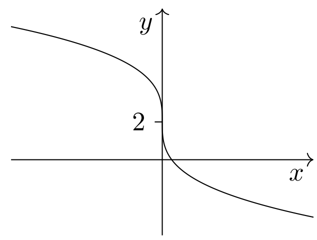
(b) If the graph has one stationary point, a local maximum at $x = \alpha$, then it is divided into two regions: the graph is increasing for $x < \alpha$ and decreasing for $x > \alpha$. In each of these regions, there can be at most one crossing of $y = 0$. Hence, $\mathrm{h}(x) = 0$ has, at most, two roots.

1375. The area of the relevant sector is $\frac{1}{2}2^2 k\pi = 2k\pi$. The area of the triangle is $\frac{1}{2}2^2 \sin k\pi = 2\sin k\pi$. So, we need to solve

$$2k\pi - 2\sin k\pi = \tfrac{5}{3}\pi - 1.$$

Since k is a rational number, we can equate the coefficients of π, giving $2k = 5/3 \Longrightarrow k = 5/6$. Checking the rational terms, $2\sin\frac{5}{6}\pi = 2\times\frac{1}{2} = 1$, and our solution works.

1376. The curve $y = \sqrt[3]{x}$ is a reflection of $y = x^3$ in the line $y = x$. It is then reflected in the x axis to $y = -\sqrt[3]{x}$ and translated by 2 units in the y direction to $y = 2 - \sqrt[3]{x}$:



1377. Eliminating z from the first two equations gives $9x - 5y = 4$, and from equations 1 and 3 gives $x + 4y = 5$. Solving these simultaneously, we get $x = 1$, $y = 1$. Substituting back in, $z = 5$.

1378. We can't use the factor theorem (over $\mathbb{R}$), because $x^2 + 1$ has no real roots. So, we must attempt the factorisation explicitly. If there were such a factorisation, then the remaining factor would be linear. So, assume, for a contradiction, that

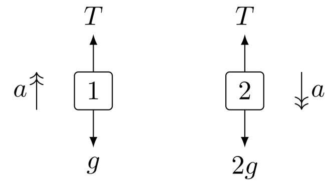
$$4x^3 - 12x^2 + 18 \equiv (x^2+1)(Ax+B).$$

Equating coefficients of x^3 tells us that $A = 1$, but then, equating coefficients of x, we get $0 = 1$. So, $(x^2 + 1)$ is not a factor.

1379. Assuming continuity of n, we find the minimum with $\frac{dA}{dn} = 6n - 85 = 0$. This yields $n = 14.16...$. Checking $n = 14$ and $n = 15$, we get $A_{14} = 398$ and $A_{15} = 400$. Hence, the lowest value in the sequence is $A_{14} = 398$.

1380. This does not hold. Since the RHS is an integral, it has an arbitrary $+c$. But the LHS doesn't: its $+c$ is eliminated by subsequent differentiation.

1381. (a) The force diagrams are



So $T-g=a$ and $2g-T=2a$. Adding, $a=\frac{1}{3}g$. This gives $1=\frac{1}{6}t^2$, so $t=\sqrt{6}$ seconds.

(b) The velocity when the string breaks is $\frac{1}{3}\sqrt{6}$. Hence, the subsequent displacement s is given by $0=\left(\frac{1}{3}\sqrt{6}\right)^2-2gs$. So, $s=0.03401...$ This is combined with the original displacement of 1 m, giving $s=1.03$ metres (3sf).

1382. Assume, for a contradiction, that there are four distinct points $x=x_i$ for $i=1,2,3,4$ on a cubic $y=ax^3+bx^2+cx+d$ that are collinear, lying on $y=px+q$. Solving these simultaneously gives $ax^3+bx^2+(c-p)x+d-q=0$. This is a cubic, so can have at most three distinct factors. However, by the factor theorem $(x-x_i)$, for $i=1,2,3,4$, are distinct factors. This is a contradiction. Hence, no four distinct points on a cubic are collinear. □

1383. Setting $\mathrm{f}(x)=x^3+3x-1$, we differentiate to get $\mathrm{f}'(x)=3x^2+3$. Hence, the N-R iteration is

$$x_{n+1}=x_n-\frac{x^3+3x-1}{3x^2+3}.$$

Putting this over a common denominator,

$$x_{n+1}=\frac{x_n(3x^2+3)-x^3-3x+1}{3x^2+3}=\frac{2x^3+1}{3x^2+3}.$$

1384. Every polynomial of odd degree must have at least one linear factor, because every polynomial graph of odd degree must cross the x axis at least once. This isn't true of polynomials of even degree, for example $y=x^2+1$.

(a) True.
(b) False.
(c) True.

1385. Subtracting the two equations gives $\mathbf{b}=-2\mathbf{i}+4\mathbf{j}$. Substituting back in, $\mathbf{a}=4\mathbf{i}+2\mathbf{j}$. The gradients of the two vectors are $m_{\mathbf{a}}=\frac{1}{2}$, $m_{\mathbf{b}}=-2$, which are negative reciprocals, so $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

1386. Adding 180° is the same as subtracting 180° on a unit circle. Hence, the identity given is the same as $\sin^2(\theta+180°)+\cos^2(\theta+180°)=1$. Let $\phi=\theta+180°$, and this is $\sin^2\phi+\cos^2\phi\equiv 1$. □

1387. The original statement is $q^x=p$. Cube rooting both sides, we get $\sqrt[3]{q^x}=\sqrt[3]{p}$. Switching the order of the multiplied indices, this gives $\left(\sqrt[3]{q}\right)^x=\sqrt[3]{p}$.

1388. (a) There is only one successful outcome, so

$$p=\tfrac{3}{6}\times\tfrac{2}{5}\times\tfrac{1}{4}=\tfrac{1}{20}.$$

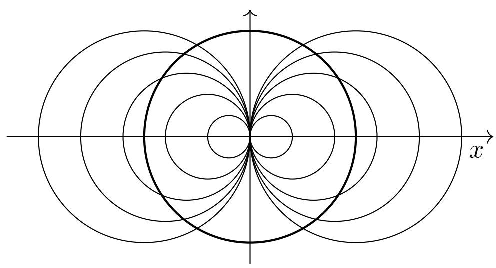
(b) There are six successful outcomes: WWB in any order and BBW in any order. Each outcome has the same probability, giving

$$p=6\times\tfrac{3}{6}\times\tfrac{2}{5}\times\tfrac{3}{4}=\tfrac{9}{10}.$$

1389. Written longhand,

$$\begin{aligned}&\frac{1}{1-x}+\frac{1}{1-x^2}=0\\ \Longrightarrow\ &\frac{1-x^2}{1-x}+1=0\\ \Longrightarrow\ &1+x+1=0\\ \Longrightarrow\ &x=-2.\end{aligned}$$

1390. The first circle is centred at $(b,0)$ and passes through the origin (radius b). Hence, if any part of the first circle is on or outside the unit circle, then they will intersect.



This occurs whenever $|b|\geq\frac{1}{2}$.

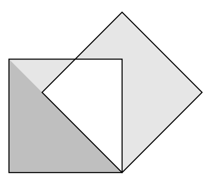
1391. The fractions available to play with are

$$\tfrac{6}{12},\tfrac{4}{12},\tfrac{3}{12},\tfrac{2}{12},\tfrac{1}{12}.$$

So, the combination needed is

$$\tfrac{5}{12}=\tfrac{3}{12}+\tfrac{2}{12}=\tfrac{1}{4}+\tfrac{1}{6}.$$

1392. We can split each shaded section as follows:



Considering the diagonal of length $\sqrt{2}$, the small triangle has side length $\sqrt{2}-1$. Hence, its area is $\frac{1}{2}(\sqrt{2}-1)^2=\frac{1}{2}\left(3-2\sqrt{2}\right)$. The darker triangle has area $\frac{1}{2}$. In total, two of each triangle makes

$$A=1+\left(3-2\sqrt{2}\right)=4-2\sqrt{2}.$$

1393. (a) Yes, at $x = -\frac{1}{2}$.

(b) No, because the $+1$ is a translation in y.

(c) Yes, at all points $x \leq -1$.

1394. This is well defined.

With domain $[-1, 1]$, the function $x \mapsto 1 - x^2$ has range $[0, 1]$, which are acceptable inputs for the square root function. Furthermore, they produce outputs in the range $[0, 1]$, making $[0, 2.7]$ a well-defined, albeit slightly bizarre, codomain.

1395. Translating, we have $\text{f}''(x) = ax + b$. If $a = b = 0$, then integrating twice gives any linear function. If $a = 0, b \neq 0$, then integrating twice gives a quadratic function. If $a \neq 0$, then integrating twice gives a cubic function. Hence, such functions f are polynomial of order at most 3.

1396. (a) The surface area is $A = 2a^2 + 4ab$. The volume is $1 = a^2 b$. So, substituting $b = a^{-2}$,

$$A = 2a^2 + 4a \times a^{-2} = 2a^2 + 4a^{-1}.$$

(b) The minimum surface area occurs when A is stationary relative to a. So, we differentiate and set the derivative $\frac{dA}{da}$ to zero:

$$\begin{aligned} &4a - 4a^{-2} = 0 \\ \Longrightarrow\ &a^3 = 1 \\ \Longrightarrow\ &a = 1. \end{aligned}$$

Substituting back in, this gives $A = 6$ m^2. We can verify that this is a minimum by checking the second derivative at $a = 1$; it has value $12 > 0$, giving a local minimum.

1397. (a) Solving $y = 4t - 4t^2 = 0$ gives $t_1 = 0$, $t_2 = 1$.

(b) $\dfrac{dx}{dt} = 2$.

(c) Substituting for y and $\dfrac{dx}{dt}$,

$$\begin{aligned} A &= \int_0^1 (4t - 4t^2) \cdot 2\, dt \\ &= \left[4t^2 - \tfrac{8}{3}t^3\right]_0^1 \\ &= \left(4 - \tfrac{8}{3}\right) - (0) \\ &= \tfrac{4}{3}. \end{aligned}$$

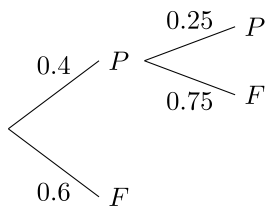
1398. This is not possible, by definition (that is, by the *modern* definition, not by Newton's original) of the word "reaction". "Reaction" and "friction" are (now) used to refer to the components of a contact force perpendicular and parallel to the surfaces in contact. Hence, they can never have a component in the same direction.

1399. Expressing x as a power of a, the LHS is

$$\begin{aligned} &x^{\log_a y} \\ \equiv\ &(a^{\log_a x})^{\log_a y} \\ \equiv\ &a^{\log_a x \log_a y}. \end{aligned}$$

Since this expression is symmetrical in x and y, the RHS must also be equal to it, proving the identity.

1400. (a) Since, $0.4 \times \mathbb{P}(\text{passing second}) = 0.1$, we know that $\mathbb{P}(\text{passing second}) = 0.25$.



(b) The possibility space is restricted to the lower two outcomes. So, the probability is

$$\frac{\mathbb{P}(\text{fail first})}{\mathbb{P}(\text{fail})} = \frac{0.6}{0.6 + 0.4 \times 0.75} = \frac{2}{3}.$$

(c) $\mathbb{P}(\text{acceptance}) = 0.1$, so $X \sim B(2, 0.1)$. Thus, $\mathbb{P}(X = 1) = {}^2\text{C}_1 \times 0.1 \times 0.9 = 0.18$.

——— End of 14th Hundred ———